Theory and applications of the Maxwell stress tensor

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The Maxwell stress tensor may be used to calculate electric and magnetic forces on objects. The method is seldom discussed in introductory texts on electromagnetism. Advanced texts often present the Maxwell stress tensor as a mathematical abstraction without explaining why is is useful. In reality, the method is essential for practical force calculations in numerical codes. In this tutorial, I will review the theory and emphasize the specific reasons why the method is used. To illustrate, I will concentrate on magnetostatic forces.

The Lorentz force density is familiar from basic electromagnetism courses:

$$\mathbf{f} = \mathbf{j} \times \mathbf{B}. \tag{1}$$

The vector quantity $\mathbf{f}$ is the force per volume in a region with magnetic flux density $\mathbf{B}$ and current density $\mathbf{j}$. The total magnetic force on an object is given by a volume integral of the force density over its volume:

$$\mathbf{F} = \int\int\int dV \; \mathbf{j} \times \mathbf{B}. \tag{2}$$

It is important to recognize that the quantity $\mathbf{j}$ is the total current density within the object, the sum of applied and atomic contributions. The applied current density arises from currents in coils. The magnitude, location and direction of the applied current density follows from the known coil geometry and drive currents. Therefore, is relatively easy to apply Eq. 2 in numerical codes to find coil forces. In contrast, volume integrals are not practical to find force contributions from magnetically-active materials (field excluders, iron and permanent magnets). In this case, the current may be concentrated in thin surface layers.

To find an alternate force expression, we start from Ampere's law:

$$\nabla \times \mathbf{B}. = \mu_0 \; \mathbf{j}. \tag{3}$$

Note that the relative magnetic permeability (μ_r) does not appear in Eq. 3 because $\mathbf{j}$ represents the sum of all currents (applied and material). Substitution in Equation 2 gives an expression for the body force entirely in terms of the magnetic flux density:

$$\mathbf{F} = \frac{1}{\mu_0} \int\int\int dV \; (\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{4}$$

Expanding the curl and the cross product, the x component of Equation 4 is

$$F_x = \frac{1}{\mu_o} \int\int\int dV \left(B_z \frac{\partial B_x}{\partial z} - B_z \frac{\partial B_z}{\partial x} - B_y \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_x}{\partial y} \right). \tag{5}$$

Consider taking the divergence of the following vector:

$$\mathbf{S}_x = \frac{1}{\mu_0} \left[\mathbf{x} \left(\frac{B_x^2 - B_y^2 - B_z^2}{2} \right) + \mathbf{y}\ B_x B_y + \mathbf{z}\ B_x B_z \right] \tag{6}$$

The result is

$$\nabla \cdot \mathbf{S}_x =$$

$$B_x \left(\frac{\partial B_x}{dx} + \frac{\partial B_y}{dy} + \frac{\partial B_z}{dz} \right) - B_y \frac{\partial B_y}{\partial x} - B_z \frac{\partial B_y}{\partial x} + B_y \frac{\partial B_x}{\partial y} + B_z \frac{\partial B_x}{\partial z}. \tag{7}$$

The condition $\nabla \cdot \mathbf{B} = 0$ implies that the terms in parenthesis sum to zero. Note that the remaining terms are identical to those in Equation 5. We can therefore write the x component of body force as

$$F_x = \frac{1}{\mu_o} \int\int\int dV\ \nabla \cdot \mathbf{S}_x. \tag{8}$$

Finally, we can apply the divergence theorem to convert the volume integral to a surface integral:

$$F_x = \frac{1}{\mu_o} \int\int dA\ \mathbf{S}_x \cdot \mathbf{n}. \tag{9}$$

The surface in Equation 9 may be any closed surface surrounding the object. The quantity $\mathbf{n}$ is a unit vector pointing out of the surface. Applying similar operations to the other force components leads to the general force law

$$\mathbf{F} = \frac{1}{\mu_o} \int\int dA\ \overline{\mathbf{S}} \cdot \mathbf{n}. \tag{10}$$

The quantity $\overline{\mathbf{S}}$ is the Maxwell stress tensor for magnetostatic fields:

$$\mathbf{S} = \frac{1}{\mu_0} \begin{bmatrix} B_x^2 - B^2/2 & B_x B_y & B_x B_z \\ B_y B_x & B_y^2 - B^2/2 & B_y B_z \\ B_z B_x & B_z B_y & B_z^2 - B^2/2 \end{bmatrix}, \tag{11}$$

where $B^2 = B_x^2 + B_y^2 + B_z^2$.

There are three reasons why Equation 11 is better suited to a numerical calculation than Equation 2:

> The accuracy of the integral is not affected by the distribution of current density within the onject. In particular, the thin surface layers of magnetically-active materials present no problem.

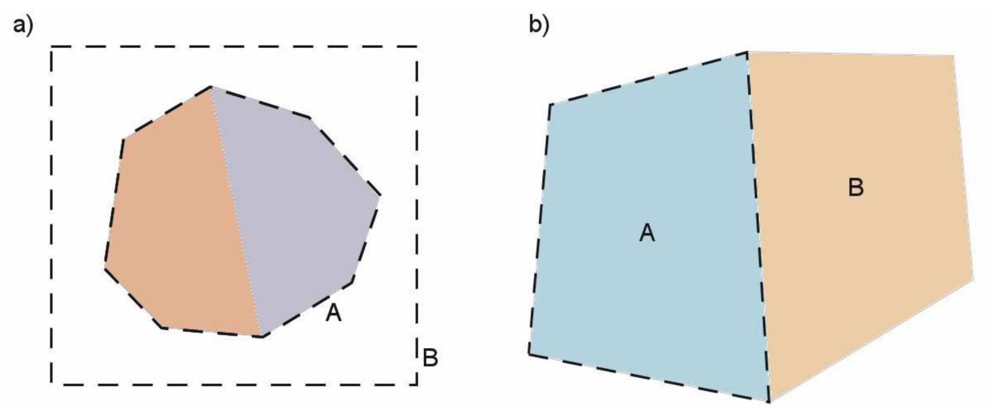

Figure 1: Surface integral of the Maxwell stress tensor. a)Valid surfaces for integrals around magnetically-active materials. A: region surfaces bounded by air elements. B:surface of an air region that encloses the magnetic materials. b)Invalid surface for an integral. The enclosed material current is undefined on the common boundary.

> The surface need not coincide with the physical surface of the object. If the object has sharp corners and regions of field enhancement, we can improve accuracy be using a diagnostic surface removed from the physical surface.
>
> The integral depends only on the field distribution outside the object. It is not necessary to know the exact current density distributions within complex anisotropic or nonlinear materials.

Some care must be exercised in applying the method. Because the integral determines the force on all applied and material currents inside the surface, we must avoid ambiguities in the enclosed current. The surface must be in an air region surrounding the object. Here, the term air implies that the region contains no material currents ($\mu = \mu_0$). Figure 1 illustrates the logic of the calculation. If we have an assembly of ferromagnetic and permanent-magnet objects surrounded by an air volume, then we can take the integral of Eq. 10 over any enclosing surface. One choice is the outer boundary of the assembly components (designated surface A in Fig. 1). In this case, we evaluate magnetic field values in the air elements near the surface to ensure that the integral encloses all material currents of the object. We could also define an arbitrary surface by enclosing the assembly inside a diagnostic air region. The integral over the surface marked B in Fig. 1 gives the same result (to within the numerical accuracy of field interpolations). Figure 1b shows a case where the stress tensor integral may not give the correct result. In this case the integral extends over the surface of ferromagnetic region A that is in contract with another iron or permanent-magnet region (B). The field values in region B along the common boundary include the effects of the surface currents of both regions. The calculation gives the force on region A plus an indeterminate portion of the force on region B.