Calculating mutual capacitance with numerical methods

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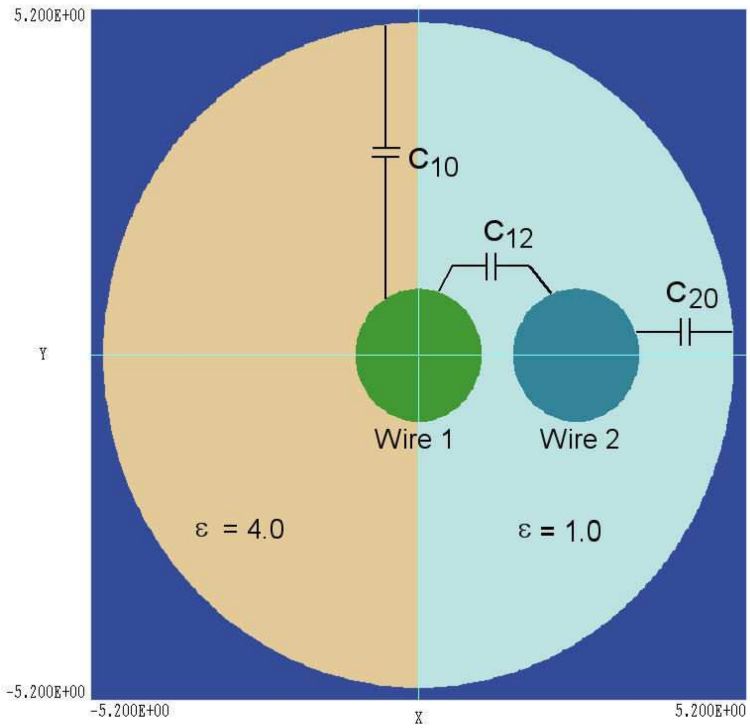

Figure 1: Benchmark example – geometry and definition of quantities (system is uniform in z with a length $L_z = 0.02$ m).

Finite-element programs for electrostatics are useful for calculating mutual capacitance in a multi-electrode system. **PhiView** has several automatic calculation routines to facilitate the process. Because the user must exercise some care to ensure accuracy and physical validity, we shall review available methods in this chapter and work through an example. Figure 1 shows a benchmark calculation that gives good accuracy and runs quickly (input files `MUTUALCAP.MIN` and `MUTUALCAP.HIN`). The geometry consists of two circular wires inside a grounded outer conductor with a circular inner radius. The enclosed volume is divided into two dielectric regions with $\epsilon_r = 4.0$ and $\epsilon_r = 1.0$. In this case, there are three unknown capacitance quantities: C_{10} (capacitance between Wire 1 and ground, C_{20} (capacitance between Wire 2 and ground) and C_{12} (mutual capacitance between Wires 1 and 2). The assembly is uniform along the z direction with length $L_z = 0.02$ m.

One approach is to employ the automatic surface integral routines in **PhiView** to calculate induced charge. To perform an integral, open a data file in the main menu and then go to the *Analysis* menu. Click on the command *Region properties* and specify the number of the region over which integrals will be performed. **PhiView** lists results on the screen and writes a record in the data file.

Table 1: Calculated quantities for example `MUTUALCAP`.

Sol	V1	V2	Q1	Q2	Q0	U
1	1.0	0.0	2.13×10^{-12}	6.09×10^{-13}	1.53×10^{-12}	1.05×10^{-12}
2	0.0	1.0	6.09×10^{-13}	1.27×10^{-12}	6.67×10^{-13}	6.27×10^{-13}
3	1.0	1.0	1.52×10^{-12}	6.64×10^{-13}	2.20×10^{-12}	1.08×10^{-12}

To illustrate, consider the capacitance quantities in the example of Fig. 1. We create Solution 1 with Wire 1 set to $V_1 = 1.0$ V and Wire 2 to $V_2 = 0.0$ V (Fig. 2). In **PhiView** we use the *Region properties* command for Regions 1 (ground electrode), 4 (Wire 1) and 5 (Wire 2). Table 1 shows the calculated surface charge values in coulombs. In the table, Q_1 is the surface charge on Wire 1, Q_2 is the charge on Wire 2, and Q_0 is the charge on the inside of the ground electrode. Because we applied a 1.0 V drive, the numbers also equal the capacitance in farads. From the table, we can determine that $C_{12} = 6.09 \times 10^{-13}$ F and $C_{10} = 1.05 \times 10^{-12}$ F. To check accuracy, we can verify that $Q_1 \cong Q_2 + Q_0$. We next generate Solution 02 with $V_1 = 0.0$ V and $V_2 = 1.0$ V. The results (shown in the second data row of Table 1) imply that $C_{12} = 6.09 \times 10^{-13}$ F and $C_{20} = 6.65 \times 10^{-13}$ F.

The surface integral method works well in the example because we have used a fine mesh for good resolution of the curved electrode surfaces. The accuracy of the method declines significantly when electrodes have sharp corners where the electric field is effectively undefined. For these situations we can get better results with volume integral methods that are less sensitive to local field interpolation errors.

For a simple two-electrode system, the volume integral of field energy is related to the capacitance by

$$U_e = \frac{CV^2}{2}. \tag{1}$$

We can use the *Full analysis* command in the *Analysis* menu of to perform a field energy integral. In response, **PhiView** lists the global energy and components associated with different regions. It is not difficult to extend this concept to multiple electrodes. For an applied voltage of 1.0 V, the global electrostatic energy U_1 for Solution 1 is related to the capacitance values by

$$U_1 = \frac{1}{2}\ (C_{12} + C_{01})\,. \tag{2}$$

Similarly, the field energy for Solution 02 is

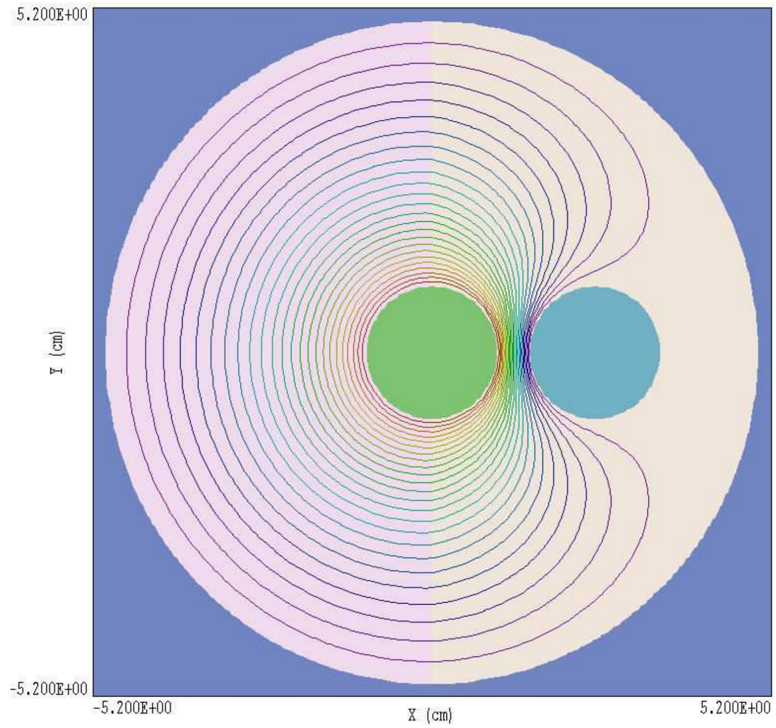

Figure 2: Calculated equipotential lines for Solution 1.

$$U_2 = \frac{1}{2}\ (C_{12} + C_{02})\,. \tag{3}$$

We perform a third solution where a 1.0 V potential is applied to both Wires 1 and 2. Here, the field energy is related to capacitance values by

$$U_3 = \frac{1}{2}\ (C_{10} + C_{30})\,. \tag{4}$$

Using the three equations, we can solve for the capacitance values in terms of the calculated values of field energy:

$$C_{12} = U_1 + U_2 - U_3, \tag{5}$$

$$C_{10} = U_1 - U_2 + U_3, \tag{6}$$

$$C_{20} = -U_1 + U_2 + U_3. \tag{7}$$

Table 1 shows field energy values for the three solutions. The calculated values of capacitance are $C_{12} = 5.98 \times 10^{-13}$ F, $C_{10} = 1.51 \times 10^{-12}$ F and $C_20 = 6.65 \times 10^{-13}$ F. These values are close to those determined by the surface integral method.

Table 2: Applied voltages for a three-electrode system with ground.

Solution	**V1**	**V2**	**V3**
1	1.0	0.0	0.0
2	0.0	1.0	0.0
3	0.0	0.0	1.0
4	1.0	1.0	0.0
5	1.0	0.0	1.0
6	0.0	1.0	1.0

There are six unknown quantities for a system with three electrodes and a ground reference: C_{12}, C_{23}, C_{31}, C_{10}, C_{20} and C_{30}. We can generate a set of six equations similar to Eqs. 7 with the set of voltages shown in Table 2.