Charged-particle beam emittance calculations

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In this tutorial I'll discuss the basic concepts of beam emittance and show how the quantity is calculated for cylindrical beams in the two-dimensional **Trak** program. I'll begin by reviewing some basic concepts of emittance. A comprehensive discussion is given in Chaps. 3 and 4 of my book **Charged Particle Beams**. It's available in PDF format on our Internet site at:

`http//www.fieldp.com/cpb.html`

A charged particle *beam* is a collection of electrons or ions where particles have about the same energy and move in about the same direction. In a perfect beam, the transverse velocities of particles are coherent so that they could be focused to a point.

The quantity *emittance* quantifies the deviation from coherency. The concept is effectively illustrated by a *phase-space* plot. Assume that the average particle motion is in the z direction and consider transverse motion in the x direction. We can characterize the transverse beam distribution at location in z with a two-dimensional plot of position x and angle with respect to the beam axis:

$$x' = \frac{dx}{dz}. \tag{1}$$

Generally, the angle is small compared to a radian for beams that are useful for applications.

The phase-space plot is a straight line for a perfect beam. The phase-space distribution of an imperfect beam has non-zero area. In principle, the distribution could have any shape. In practice, we often assume the shape is elliptical because particles follow elliptical trajectories in phase space as the beam moves along z in a focusing system with linear transverse forces. The upright ellipse of Fig. 1 shows a beam at a waist point. The phase space ellipses of diverging and converging beams are inclined with respect to the x and x' axes. If the ellipse of the figure encloses a uniform density of particle points, then the emittance is defined as the area of the ellipse divided by π, or

$$\epsilon_x = x_0\ x'_0. \tag{2}$$

If the value of x_0 is specified in meters and x'_o in radians, then the unit of emittance is π-m-radian.

In the more general case, the distribution shape may not be elliptical, the density may not be uniform and the beam may not be at a waist point. Here, it is useful to use the RMS (root-mean-squared) emittance, given by the equation:

$$\epsilon_x = 4\ \left[\overline{(x-\overline{x})^2}\ \overline{(x'-\overline{x'})^2} - \overline{(x-\overline{x})(x'-\overline{x'})}^2\right], \tag{3}$$

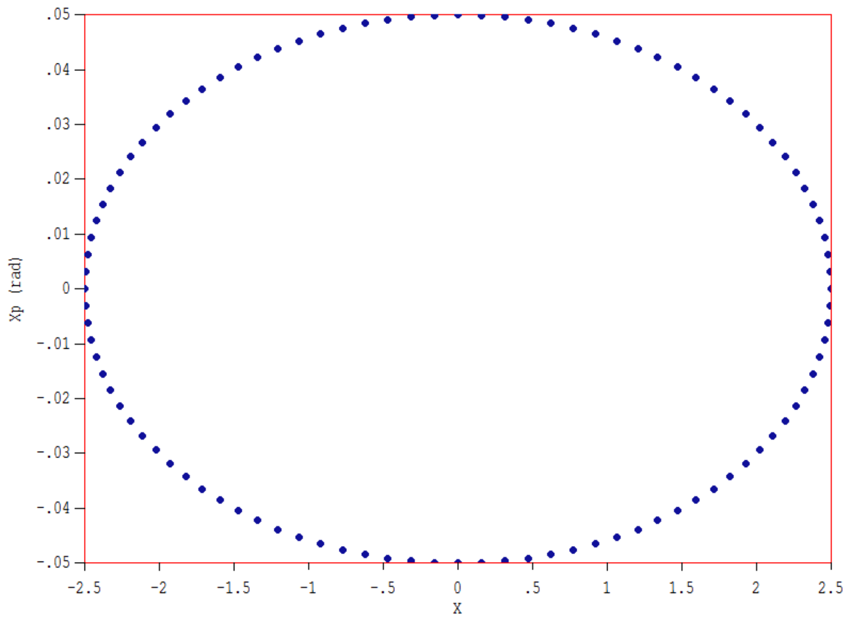

Figure 1: Elliptical phase-space distribution with $x_0 = 2.5$ cm and $x'_0 = 0.05$ radian.

where the overline symbol represents an average taken over the particle distribution. The equation applies to general distributions (e.g., converging and diverging beams). The factor of 4.0 ensures that the equation gives the result of Eq. 2 when applied to a uniformly-filled, upright ellipse. Equation 3 is employed for emittance calculations in **Trak** and **Gendist**.

The example `EMITCALC` illustrates **Trak** emittance figures. An accelerating electric field and focusing magnetic field with cylindrical symmetry may be included in the solution. The electric field volume defined by `EMITCALCE.MIN` extends axially from $z = 0.0$ cm to $z = 100.0$ cm and radially from $r = 0.0$ cm to $r = 10.0$. Fixed potential boundaries with $\phi = 0.0$ V at $z = 0.0$ cm and $\phi = 1.0$ V at $z = 100.0$ cm give a uniform electric field $E_z = 10.0$ V/m. The magnetic field calculation (shown in Fig. 2) extends radially to 20.0 cm. It represents a solenoid magnetic lens. The drive current of 1000 A gives a peak field $B_z = 2.4967 \times 10^{-3}$ tesla. It is important to understand the role of model particles in cylindrical calculations with **Trak**. In an electron beam simulation, the program tracks model particles as individual electrons in Cartesian coordinates. An electron occupies a single value of azimuth at each axial position. On the other hand, a model particle is treated as if it extended over 360° of azimuth for the assignment of space-charge and current to calculate beam-generated fields. We must exercise some care in interpreting emittance calculations.

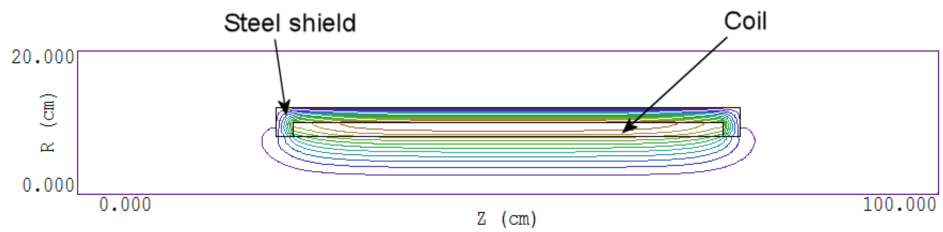

Figure 2: Magnetic field solution `EMITCALCB`.

When **Trak** generates model particles at an emission surface, the convention is place them along the x axis ($\theta = 0.0^o$). To represent this choice, we shall start the `EMITCALC` example with list input that defines a distribution with non-zero emittance such that particles have initial positions and angles along x. The spreadsheet `EMITCALC.XLS` creates a distribution of 100 electrons with kinetic energy 250 keV distributed uniformly around the periphery of a phase-space ellipse with dimensions $x_0 = 2.5$ cm and $x'_0 = 0.05$ radian (Fig. 1). A uniformly-filled ellipse with this boundary would have an emittance 1.25×10^{-3} π-m-rad (Eq. 2). We copy the particle information in the final nine columns of the spreadsheet to the text file `EMITCALC.PRT`. Table 1 shows the **Trak** input file. In the *Fields* section, we can use the normalization factors in the *EFile* and *BFile* commands to turn off the electric and/or magnetic field or to change the magnitude. The *Record* command generates output `PRT` files at positions $z = 25.0, 50.0$ and 75.0 cm in addition to the normal output `EMITCALCOUT.PRT` at the end of the solution volume ($z = 100.0$ cm). The *PartDist* command write listing-file data that include emittance calculations in x and y using Eq. 3.

We now have sufficient resources to begin calculations. In the first run, we set both field normalization factors equal to 0.0. In this case the beam simply drifts so that the output distribution is an expanding and tilted version of the input (diverging beam). The listing generated by the *PartDist* command contains the following entry:

```
Model particle RMS emittances with respect to the z-axis
   EpsiX (pi-m-rad):   2.5008E-03
   EpsiY (pi-m-rad):   0.0000E+00
```

Note that the calculated value of ϵ_x is twice the product $x_0 x'_0$ because the particles are distributed on the envelope of the ellipse rather than uniformly distributed through the area.

An important question is how does the emittance of an azimuthally-symmetric beam compare to the value when particles are confined to the x axis? We can use **GenDist** to resolve the issue. Load `EMITCALCIN.PRT` into the program and click the command *Beam analysis* in the *Analysis*

Table 1: **Trak** input file `EMITCALC.TIN`.

```
FIELDS
  EFILE: EmitCalcE.EOU 0.0
  BFILE: EmitCalcB.POU 6.0
  DUNIT:   1.0000E+02
END
PARTICLES TRACK
  PFILE: EmitCalcIn
  RECORD UP Z 25.0 50.0 75.0
END
DIAGNOSTICS
  PARTFILE: EmitCalcOut
  PARTDIST
END
ENDFILE
```

menu. The resulting output listing show the results $\epsilon_x = 0.25008$ π-cm-rad and $\epsilon_y = 0.0$ π-cm-rad. Next, use the *Beam section tool* to rotate and replicate particle entries to create a full circular beam. Use the default axis of z, pick the date type $2D$ and accept the default of $N_\theta = 60$. Save the results as `EMITCALCIN_COMPLETE.PRT`. When loaded in **GenDist**, the distribution gives the projection plot shown in Fig. 3. An application of the *Beam analysis* command gives the results $\epsilon_x = 0.12506$ π-cm-rad and $\epsilon_y = 0.12506$ π-cm-rad. As expected, the emittance values are equal in x and y. The effect of including particles with positions and angles off the x axis was to reduce the RMS emittance by a factor of 2.0.

To finalize our understanding of emittance values created by **Trak**, we must include effects of rotation in solenoid magnetic fields that mix positions and angles in x and y. We run a solution with the electric field normalization factor set to 0.0 and the magnetic field factor set to 4.8. The resulting magnetic field magnitude gives a rotation of about 90° passing through the lens. Table 2 lists emittance values calculated with **GenDist** using the initial and final `PRT` files as well as those generated by the *Record* command. The sum of ϵ_x and ϵ_y is conserved to high accuracy and is unaffected by the ordered rotational motion in the magnetic field.

The test calculations suggest the following procedure to find emittance for an azimuthally-symmetric beam from the RMS values calculated for a distribution of model particles (where each particle occupies a single azimuth):

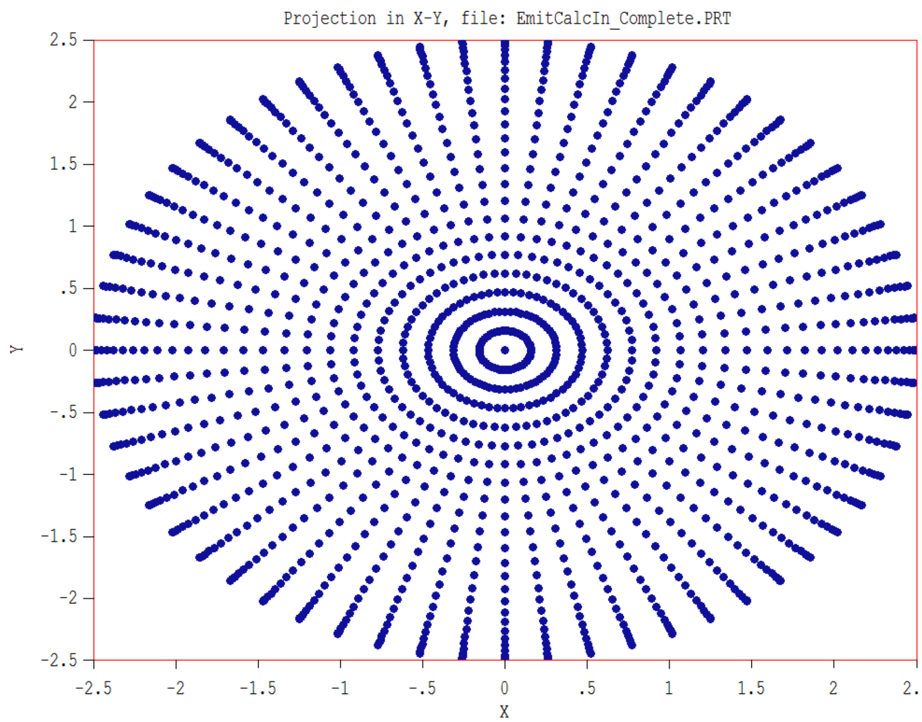

Figure 3: Azimuthally symmetric distribution created with the **GenDist** *Beam section tool.*

Table 2: Emittance calculations for a 250 keV electron beam moving through a magnetic lens with $B_0 = 0.0120$ tesla.

z (cm)	ϵ_x (π-cm-rad)	ϵ_y (π-cm-rad)	$\epsilon_x + \epsilon_y$ (π-cm-rad)
0.0	0.25008	0.00000	0.25008
25.0	0.24897	0.00113	0.25010
50.0	0.12262	0.12761	0.25023
75.0	0.00075	0.24981	0.25056
100.0	0.00010	0.25012	0.25022

1. Take the sum of ϵ_x and ϵ_y.

2. Divide the sum equally between x and y.

These operations are performed automatically by **Trak** in response to the *PartDist* command for runs where the fields have cylindrical symmetry. Here is the resulting listing for the example:

```
Cylindrical beam RMS emittances with respect to the z-axis
   EpsiX (pi-m-rad):   1.2511E-03
   EpsiY (pi-m-rad):   1.2511E-03
```

To conclude, we can use the test example to verify conservation of normalized emittance:

$$\epsilon_{nx} = \beta\gamma\epsilon_x. \tag{4}$$

In this case, we use a normalization factor of 0.0 for the magnetic field and 2.5×10^5 for the electric field. For this choice, the electron exit energy is 500 keV. The initial relativistic factors are $\beta_i = 0.7410$ and $\gamma_i = 1.4892$ and the final factors are $\beta_i = 0.8629$ and $\gamma_f = 1.9785$. If normalized emittance is conserved, then the final beam emittance should be

$$\epsilon_{xf} = \epsilon_{xi}\ \frac{\beta_i\gamma_i}{\beta_f\gamma_f}. \tag{5}$$

The value at $z = 100.0$ cm reported by the *PartDist* command agrees with the theoretical prediction of 1.616×10^{-3} π-m-rad.