Emittance calculations for circular beams

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Emittance concepts

Emittance characterizes the degree of disorder in a charged particle beam. The *transverse emittance* parametrizes the thermal spread of transverse momentum. Random motion results in an effective pressure that determines focusing forces necessary to confine a beam as well as the minimum spot size of a focused beam. The text **Charged Particle Beams**[1] gives a comprehensive review of definitions and applications of emittance.

High-energy accelerators employ quadrupole lenses and bending magnets. The convention is to designate the local beam axis as z, with transverse motion in x (bending plane) and y (vertical plane). In such a system, particle motions in x and y are decoupled. In this case, we can define separate emittance values ϵ_x and ϵ_y. To make an ideal emittance calculation, we make a plot of x (displacement from the main axis) and $x' = dx/dz$ (angle with respect to the main axis) for the particle distribution at a given location in z. The quantity ϵ_x is the trace-space area of the minimal ellipse that encloses the distribution divided by π. An ideal distribution at a beam waist is one that uniformly fills ellipses with dimensions x_0-x'_0 and y_0-y'_0. In this case, the emittances are

$$\epsilon_x = x_0 \; x'_0, \tag{1}$$

$$\epsilon_y = y_0 \; y'_0. \tag{2}$$

Generally, the non-uniform trace-space distributions of real beams add ambiguity to the definition of the bounding ellipse. A resolution is to employ the RMS (root-mean-squared) expressions of Lapostolle[2]:

$$\overline{\epsilon_x} = 4.0 \sqrt{\overline{x^2} \; \overline{x'^2} - \overline{x \; x'}^2}, \tag{3}$$

$$\overline{\epsilon_y} = 4.0 \sqrt{\overline{y^2} \; \overline{y'^2} - \overline{y \; y'}^2}. \tag{4}$$

The overline symbols on the right-hand side denote averages over the particle distribution. Equations 3 and 4 hold for tilted distributions (diverging or converging beam) as well as upright distributions. The factor of 4.0 is included to ensure that the expressions give the same values as Eqs. 1 and 2 for a uniform distribution inside an upright ellipse.

Previous versions of the **Trak** and **GenDist** programs reported values of $\overline{\epsilon_x}$ and $\overline{\epsilon_y}$. These numbers could be misleading in cylindrical-beam calculations for two reasons:

[1] S. Humphries, **Charged Particle Beams** (Wiley-Interscience, New York, 1990), Chaps. 3 and 4 (available for download at http://www.fieldp.com/cpb.html)

[2] P. Lapostolle, IEEE Trans. Nucl. Sci. **NS-18**, 1101 (1971).

Model particles in **Trak** that occupy a single position in the x-y plane carry an annulus of space-charge and current to model a circular beam. In other words, the code uses a model-particle distribution that is generally not azimuthally symmetric.

Particle motions in x and y are coupled in cylindrical calculations with solenoid lenses.

Because of the first issue, $\overline{\epsilon_x}$ may not equal $\overline{\epsilon_y}$, even though the quantities should be equal for an azimuthally-symmetric beam. The second issue mixes an ordered azimuthal motion into the calculation of thermal velocities, leading to meaningless emittance values.

Following a suggestion of Carl Ekdahl of Los Alamos National Laboratory, I added a third calculated RMS emittance quantity to **Trak** and **GenDist** that applies specifically to circular beams. The following formula was adapted from the theory of Lee and Cooper[3]:

$$\overline{\epsilon_r} = 2.0 \sqrt{\overline{r^2} \left[\overline{(p_r/p_z)^2} + \overline{(p_\theta/p_z)^2} \right] - \left(\overline{rp_r/p_z}\right)^2 - \left(\overline{rp_\theta/p_z}\right)^2}, \tag{5}$$

where

$$r = \sqrt{x^2 + y^2}. \tag{6}$$

For a particle at azimuth θ, the radial and azimuthal momenta are given in terms of the Cartesian components by

$$p_r = p_x \cos(\theta) + p_y \sin(\theta), \tag{7}$$

$$p_\theta = -p_x \sin(\theta) + p_y \cos(\theta). \tag{8}$$

Equation 5 gives a correct value of $\overline{\epsilon_r}$ for use in the paraxial ray equation for a circular beam. The quantity is invariant in a solenoid lens. Furthermore, with a valid beam distribution the value does not require a cylindrically-symmetric model-particle distribution.

Program modifications

Trak calculates an assortment of statistical quantities in response to the `PARTDIST` command in the `DIAGNOSTICS` section and records the results in the listing file (`TLS`). The program uses particle properties at the end of their trajectories. This listing has the following form:

[3] E. Lee and R. Cooper, Particle Accelerators **7**, 83 (1976).

```
     Final Particle Distribution
Average position along z-axis (meters):  9.999926E-01
RMS displacements from the z-axis
   X (meters):  3.892413E-03
   Y (meters):  1.696705E-02
   R (meters):  1.740781E-02
RMS angles with respect to the z-axis
   Xp (radians):  3.164028E-02
   Yp (radians):  6.494762E-02
   Rp (radians):  6.629889E-02
RMS emittances with respect to the z-axis
   EpsiX (pi-m-rad):  4.550165E-04
   EpsiY (pi-m-rad):  1.920744E-03
   EpsiR (pi-m-rad):  1.417018E-03
Maximum displacement from the z-axis
   RMax (meters):  2.567633E-02
Magnitude of the maximum particle angle with respect to the zaxis
   RpMax (radians):  1.341721E-01
```

Note that SI units are used and that quantities are calculated relative to the z axis. The quantities *EpsiX* and *EpsiY* are calculated from Eq. 3 and 4 and the new quantity *EpsiR* is determined from Eq. 5.

GenDist determines statistical quantities using PRT files as input. Table 1 shows an example of output in response to the *Beam analysis* command. There are three differences from the **Trak** analysis:

> Calculations may be performed for all `PRT` files created in a run.
>
> Length dimensions are those used in the **Trak** simulation. For example, with $DUnit = 100.0$, emittance dimensions are π-cm-rad.
>
> Calculations may be performed with respect to the x, y or z axes. As an example, for reference axis y the radial quantity used in Eq. 5 is $r = \sqrt{z^2 + x^2}$.

Test calculations

I set up the **Trak** simulation `EMITCALC` to check the validity of the new routines and to demonstrate how to use **GenDist** to create input circular beams with non-zero emittance. Figure 1 shows the geometry. The solution volume had outer radius 20.0 cm and axial length 100.0 cm. A low-current beam electron beam was injected through the left-hand boundary. The beam, with radius 2.5 cm and kinetic energy 250 keV, passed through a solenoid lens. The **PerMag** solution for the lens had a peak field of 25.0 G. A

Table 1: **GenDist** statistical report

```
Beam distribution analysis for file emitcalc03.PRT
     Reference axis: Z
         X(origin):  0.0000E+00
         Y(origin):  0.0000E+00

Averaged quantities in absolute space
  Total number of entries in file:     3000
  Average position
     X:  9.0926E-03
     Y:  1.6374E-03
     Z:  7.5000E+01
  Average direction vector
     Ux: -0.0003853
     Uy:  0.0001055
     Uz:  0.9972631
  Average energy:  2.5000E+05 (eV)

Averaged quantities relative to the reference axis
  Average displacement
     X-XRef:  9.0926E-03
     Y-YRef:  1.6374E-03
  Average angle (degrees)
     ThetaX:   -0.022
     ThetaY:    0.006
  RMS angular spread (degrees)
     dThetaX:    2.894
     dThetaY:    3.104
  RMS position spread relative to reference axis
     dX:  7.3704E-01
     dY:  7.5249E-01
     dR:  1.0533E+00
  RMS position spread relative to average position
     dX:  7.3698E-01
     dY:  7.5248E-01
     dR:  1.0533E+00
  RMS emittance
     EpsiX:  1.3764E-01
     EpsiY:  1.5474E-01
     EpsiR:  1.4037E-01
  RMS energy spread:  3.3975E+00 (eV)
```

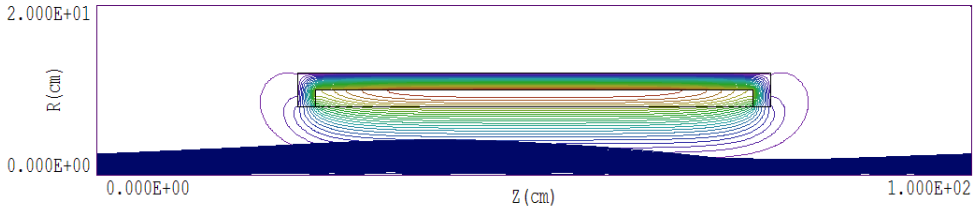

Figure 1: Example EMITCALC: z-r plot of magnetic field lines and model-particle trajectories.

normalization factor of 4.8 was introduced in the **Trak** simulation to give a beam rotation of about 90°. I included an option for a uniform axial electric field to check conservation of normalized momentum. The electric field was set to zero for the reported calculations.

I prepared two **GenDist** input files to create 1) a full circular beam (EMITCALCFULL) and 2) a line beam along the x axis (EMITCALCLINE). Table 2 shows the file to create the full beam. The *Def(Circ)* command created a circular beam moving in z with radius 2.5 cm and 2500 model particles distributed uniformly in azimuth. The weighting function defined in the *RDist* structure ensured an approximately uniform density of particles in configuration space. The *DxDist* and *DyDist* commands introduced a uniform distribution in transverse angles between ± 0.05 radians at all positions. The quantity $\overline{\epsilon_r}$ should be roughly equal to 0.125 π-cm-rad.

The **Trak** input file EMITCALC.TIN has the content

```
FIELDS
  EFILE: EmitCalcE.EOU 0.0
  BFILE: EmitCalcB.POU 4.8
  DUNIT:   1.0000E+02
END
PARTICLES TRACK
  PFILE: EmitCalcFull
  RECORD UP Z 25.0 50.0 75.0
END
DIAGNOSTICS
  PARTLIST
  PARTFILE: EmitCalcOut
  PARTDIST
  BSCAN 0.0 0.0 0.0 0.0 0.0 100.0
END
ENDFILE
```

Note the use of the RECORDPLANE command. With inclusion of the input file EMITCALCFULL.PRT and output file EMITCALCOUT.PRT, particle files were

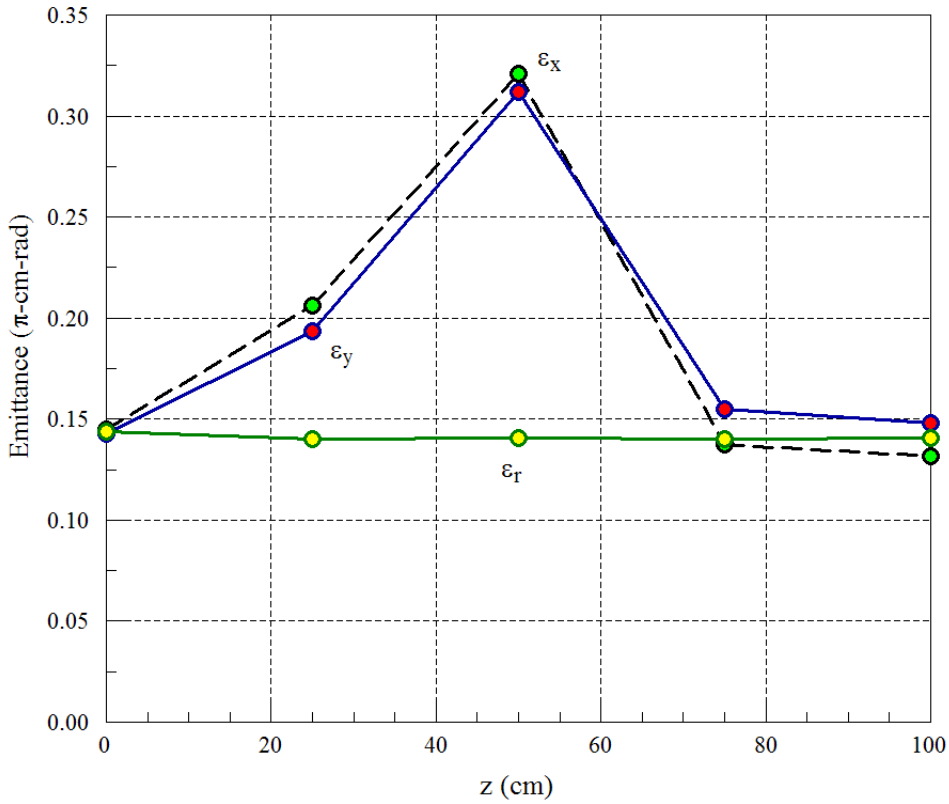

Figure 2: `EMITCALC` example, azimuthally-symmetric beam. Emittance calculations using **GenDist** at different axial positions.

available at $z = 0.0$, 25.0, 50.0, 75.0 and 100.0 cm. Figure 1 shows the envelope of the high-emittance beam. After expanding from a waist, it was focused to a divergence-limited spot by the lens.

Figure 2 shows calculated values of ϵ_x, ϵ_y and ϵ_r at different axial positions. The condition $\epsilon_x \cong \epsilon_y \cong \epsilon_r$ holds outside the region of the lens field. There were large errors in ϵ_x and ϵ_y in the region where the beam rotates. In contrast, ϵ_r had the fixed value 0.140 π-cm-rad throughout the simulation volume.

Next, I prepared an equivalent beam distribution with all particles aligned along x (the typical input condition for a **Trak** simulation). Table 3 shows the **GenDist** file. In this case, the source shape was rectangular with zero width along y. The distribution had one particle along y and 500 particles along x. There was an interesting subtlety in assigning weighting in the x direction. For spatial uniformity, the weighting shoud follow $p(x) = |x|$. If the rectangular source filled the range -2.50 cm $\leq x \leq$ 2.50 cm, the weighting function would be discontinuous. Instead, I set the source dimensions to -1.25 cm $\leq x \leq$ 1.25 cm, added a shift of 1.25 cm in the x direction, and used

Table 2: Contents of the `EMITCALCFULL.DST`, **GenDist** input file

```
FileType = PRT
RestMass =   0.0000E+00
Charge =  -1.0000E+00
Energy =   2.5000E+05
Def(Circ) =   0.0000E+00  2.5000E+00     50     50
Shift =   0.0000E+00  0.0000E+00  1.0000E-03
Distribution = Uniform
RDist
  0.00  0.00
  0.20  0.20
  0.40  0.40
  0.60  0.60
  0.80  0.80
  1.00  1.00
End
DxDist
  -2.865  1.000
  -2.000  1.000
  -1.000  1.000
   0.000  1.000
   1.000  1.000
   2.000  1.000
   2.865  1.000
End
DyDist
  -2.865  1.000
  -2.000  1.000
  -1.000  1.000
   0.000  1.000
   1.000  1.000
   2.000  1.000
   2.865  1.000
End
EndFile
```

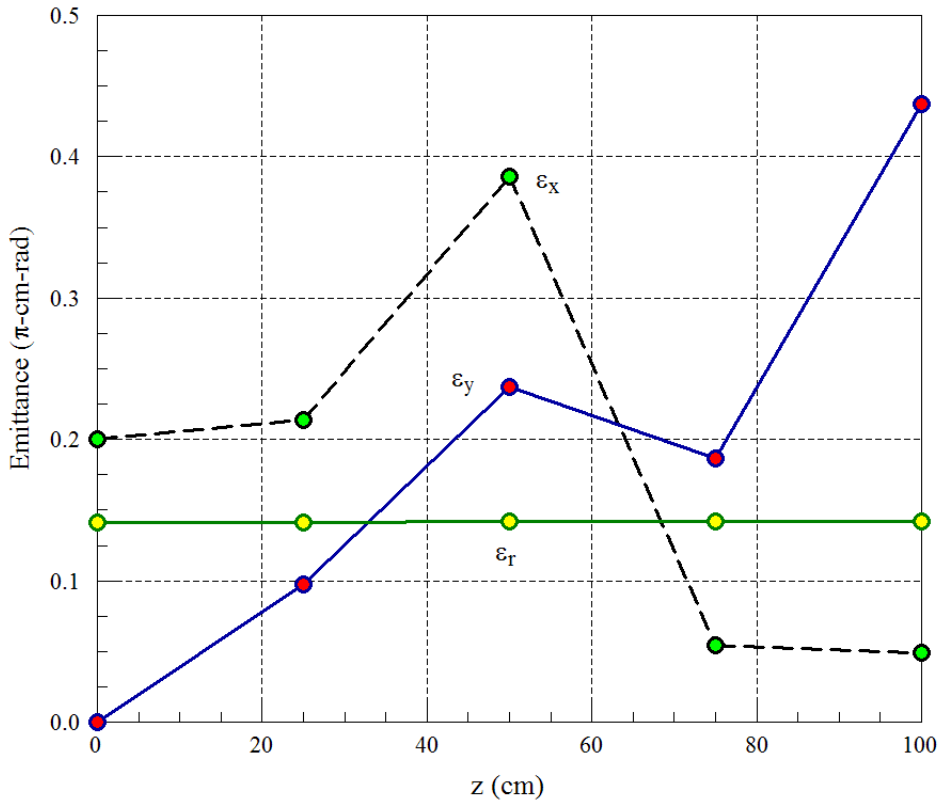

Figure 3: `EMITCALC` example, input beam aligned along x. Emittance calculations using **GenDist** at different axial positions.

the spatial weighting function $p(x) = x$. The angular distribution in x and y was the same as that for the full beam.

Figure 3 shows the calculated emittances. The quantities ϵ_x and ϵ_y were effectively meaningless over the full solution space. In contrast, ϵ_r was conserved through the lens and had the same value (0.140 π-cm-rad) as that for the azimuthally-symmetric beam. In conclusion, the Lee-Cooper emittance expression provides a reliable indicator of radial emittance in **Trak** simulations of circular beams for realistic beam distributions. By realistic, I mean that the beam varies smoothly in space and has an isotropic distribution in angle. Some input distributions lead to unexpected values of ϵ_r because they violate the basis of the RMS emittance definition. The user should avoid singular distributions, such as particles on the boundary of a phase ellipse or a beam with divergence only in one transverse direction.

Table 3: Contents of `EMITCALCLINE.DST`, **GenDist** input file

```
FileType = PRT
RestMass =   0.0000E+00
Charge =  -1.0000E+00
Energy =   2.5000E+05
Def(Rect) =   1.25000E+00 0.0000E+00  500  1
Shift =   1.2500E+00  0.0000E+00  1.0000E-03
Distribution = Uniform
XDist
  0.00  0.00
  0.20  0.20
  0.40  0.40
  0.60  0.60
  0.80  0.80
  1.00  1.00
End
DxDist
  -2.865  1.000
  -2.000  1.000
  -1.000  1.000
   0.000  1.000
   1.000  1.000
   2.000  1.000
   2.865  1.000
End
DyDist
  -2.865  1.000
  -2.000  1.000
  -1.000  1.000
   0.000  1.000
   1.000  1.000
   2.000  1.000
   2.865  1.000
End
EndFile
```