Tutorial: Surface integral expressions for electric/magnetic force and torque

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The **AMaze** and **TriComp** analysis programs include powerful routines to compute surface integrals of vector quantities over the boundaries between internal and external region sets. For a general vector field $\mathbf{A}$, a surface integral is defined by

$$\int\int_S dA\ \mathbf{A}\cdot\mathbf{n}, \tag{1}$$

where dA is a differential area on the surface and $\mathbf{n}$ is a unit vector normal to the surface pointing from the internal to the external set. The tutorial *Theory and applications of the Maxwell stress tensor*[1] discusses the importance of surface integrals in numerical codes to find forces on magnetically-active objects. In this case, the force components are given by the expressions

$$\begin{aligned} F_x &= \int\int_S dA\ \left(S_{11}n_x + S_{12}n_y + S_{13}n_z\right), \\ F_y &= \int\int_S dA\ \left(S_{21}n_x + S_{22}n_y + S_{23}n_z\right), \\ F_z &= \int\int_S dA\ \left(S_{31}n_x + S_{32}n_y + S_{33}n_z\right). \end{aligned} \tag{2}$$

The quantities S_{ij} are components of the Maxwell stress tensor. We can combine the component expressions into the succinct form

$$\mathbf{F} = \int\int_S dA\ \overline{\mathbf{S}}\cdot\mathbf{n}, \tag{3}$$

where the quantity $\overline{\mathbf{S}}$ is the tensor

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}. \tag{4}$$

For electrostatic forces, the Maxwell stress tensor has the form

$$\overline{\mathbf{S}} = \epsilon_0 \begin{bmatrix} (E_x^2 - E^2/2) & (E_xE_y) & (E_xE_z) \\ (E_yE_x) & (E_y^2 - E^2/2) & (E_yE_z) \\ (E_zE_x) & (E_zE_y) & (E_z^2 - E^2/2) \end{bmatrix}. \tag{5}$$

where $E^2 = E_x^2 + E_y^2 + E_z^2$. The tensor for magnetostatic forces is

$$\overline{\mathbf{S}} = \frac{1}{\mu_0} \begin{bmatrix} (B_x^2 - B^2/2) & (B_xB_y) & (B_xB_z) \\ (B_yB_x) & (B_y^2 - B^2/2) & (B_yB_z) \\ (B_zB_x) & (B_zB_y) & (B_z^2 - B^2/2) \end{bmatrix}. \tag{6}$$

with $B^2 = B_x^2 + B_y^2 + B_z^2$.

[1]Available for download at http://www.fieldp.com/documents/stresstensor.pdf

A differential element of torque is defined by

$$d\mathbf{t} = \mathbf{r} \times \mathbf{dF}. \tag{7}$$

Here, the vector $\mathbf{r}$ points from a torque origin $[x_t, y_t, z_t]$ to the current position:

$$\mathbf{r} = [r_x, r_y, r_z] = [(x - x_t), (y - y_t), (z - z_t)]. \tag{8}$$

The total torque resulting from the force on a body may be written in terms of the Maxwell stress tensor as:

$$\mathbf{t} = \int\int_S dA \ \mathbf{r} \times (\overline{\mathbf{S}} \cdot \mathbf{n}). \tag{9}$$

To employ the surface integral capabilities of the **AMaze** and **TriComp** programs, we need to determine a torque tensor $\overline{\mathbf{T}}$ such that the torque vector is given by an expression of the form,

$$\mathbf{t} = \int\int_S dA \ \overline{\mathbf{T}} \cdot \mathbf{n}. \tag{10}$$

We can compute $\overline{\mathbf{T}}$ by expanding the right-hand side of Eq. 9 in component form and collecting terms with common factors of n_x, n_y and n_z. The procedure gives the following form for the torque on a body in terms of an integral over a surrounding surface:

$$\begin{aligned} t_x &= \int\int_S dA \ \ (T_{11}n_x + T_{12}n_y + T_{13}n_z) \\ t_y &= \int\int_S dA \ \ (T_{21}n_x + T_{22}n_y + T_{23}n_z) \\ t_z &= \int\int_S dA \ \ (T_{31}n_x + T_{32}n_y + T_{33}n_z)\,. \end{aligned} \tag{11}$$

The components of the torque tensor are related to the components of the Maxwell stress tensor and the components of the vector from the torque origin:

$$\overline{\mathbf{T}} = \left[\begin{array}{ccc} (r_yS_{31} - r_zS_{21}) & (r_yS_{32} - r_zS_{22}) & (r_yS_{33} - r_zS_{23}) \\ (r_zS_{11} - r_xS_{31}) & (r_zS_{12} - r_xS_{32}) & (r_zS_{13} - r_xS_{33}) \\ (r_xS_{21} - r_yS_{11}) & (r_xS_{22} - r_yS_{12}) & (r_xS_{23} - r_yS_{13}) \end{array} \right]. \tag{12}$$