



Tutorial: emittance calculations for circular beams

Stanley Humphries, Copyright 2010

Field Precision

PO Box 13595, Albuquerque, NM 87192 U.S.A.

Telephone: +1-505-220-3975

Fax: +1-617-752-9077

E mail: techinfo@fieldp.com

Internet: <http://www.fieldp.com>

Emittance concepts

Emittance characterizes the degree of disorder in a charged particle beam. The *transverse emittance* parametrizes the thermal spread of transverse momentum. Random motion results in an effective pressure that determines focusing forces necessary to confine a beam as well as the minimum spot size of a focused beam. The text **Charged Particle Beams**¹ gives a comprehensive review of definitions and applications of emittance.

High-energy accelerators employ quadrupole lenses and bending magnets. The convention is to designate the local beam axis as z , with transverse motion in x (bending plane) and y (vertical plane). In such a system, particle motions in x and y are decoupled. In this case, we can define separate emittance values ϵ_x and ϵ_y . To make an ideal emittance calculation, we make a plot of x (displacement from the main axis) and $x' = dx/dz$ (angle with respect to the main axis) for the particle distribution at a given location in z . The quantity ϵ_x is the trace-space area of the minimal ellipse that encloses the distribution divided by π . An ideal distribution at a beam waist is one that uniformly fills ellipses with dimensions x_0 - x'_0 and y_0 - y'_0 . In this case, the emittances are

$$\epsilon_x = x_0 x'_0, \quad (1)$$

$$\epsilon_y = y_0 y'_0. \quad (2)$$

Generally, the non-uniform trace-space distributions of real beams adds ambiguity to the definition of the bounding ellipse. A resolution is to employ the RMS (root-mean-squared) expressions of Lapostolle²:

$$\overline{\epsilon_x} = 4.0 \sqrt{\overline{x^2 x'^2} - \overline{x} \overline{x'^2}}, \quad (3)$$

$$\overline{\epsilon_y} = 4.0 \sqrt{\overline{y^2 y'^2} - \overline{y} \overline{y'^2}}. \quad (4)$$

The overline symbols on the right-hand side denote averages over the particle distribution. Equations 3 and 4 hold for tilted distributions (diverging or converging beam) as well as upright distributions. The factor of 4.0 is included to ensure that the expressions give the same values as Eqs. 1 and 2 for a uniform distribution inside an upright ellipse.

Previous versions of the **Trak** and **GenDist** programs reported values of $\overline{\epsilon_x}$ and $\overline{\epsilon_y}$. These numbers could be misleading in cylindrical-beam calculations for two reasons:

¹S. Humphries, **Charged Particle Beams** (Wiley-Interscience, New York, 1990), Chaps. 3 and 4 (available for download at <http://www.fieldp.com/cpb.html>)

²P. Lapostolle, IEEE Trans. Nucl. Sci. **NS-18**, 1101 (1971).

- Model particles in **Trak** that occupy a single position in the x - y plane carry an annulus of space-charge and current to model a circular beam. In other words, the code uses a model-particle distribution that is generally not azimuthally symmetric.
- Particle motions in x and y are coupled in cylindrical calculations with solenoid lenses.

Because of the first issue, $\overline{\epsilon_x}$ may not equal $\overline{\epsilon_y}$, even though the quantities should be equal for an azimuthally-symmetric beam. The second issue mixes an ordered azimuthal motion into the calculation of thermal velocities, leading to meaningless emittance values.

Following a suggestion of Carl Ekdahl of Los Alamos National Laboratory, I added a third calculated RMS emittance quantity to **Trak** and **GenDist** that applies specifically to circular beams. The following formula was adapted from the theory of Lee and Cooper³:

$$\overline{\epsilon_r} = 2.0 \sqrt{r^2 \left[\overline{(p_r/p_z)^2} + \overline{(p_\theta/p_z)^2} \right] - \left(\overline{rp_r/p_z} \right)^2 - \left(\overline{rp_\theta/p_z} \right)^2}, \quad (5)$$

where

$$r = \sqrt{x^2 + y^2}. \quad (6)$$

For a particle at azimuth θ , the radial and azimuthal momenta are given in terms of the Cartesian components by

$$p_r = p_x \cos(\theta) + p_y \sin(\theta), \quad (7)$$

$$p_\theta = -p_x \sin(\theta) + p_y \cos(\theta), \quad (8)$$

$$(9)$$

Equation 5 gives a correct value of $\overline{\epsilon_r}$ for use in the paraxial ray equation for a circular beam. The quantity is invariant in a solenoid lens. Furthermore, with a valid beam distribution the value does not require a cylindrically-symmetric model-particle distribution.

Program modifications

Trak calculates an assortment of statistical quantities in response to the PARTDIST command in the DIAGNOSTICS section and records the results in the listing file (TLS). The program uses particle properties at the end of their trajectories. This listing has the following form:

³E. Lee and R. Cooper, Particle Accelerators **7**, 83 (1976).

```

Final Particle Distribution
Average position along z-axis (meters):  9.999926E-01
RMS displacements from the z-axis
  X (meters):  3.892413E-03
  Y (meters):  1.696705E-02
  R (meters):  1.740781E-02
RMS angles with respect to the z-axis
  Xp (radians):  3.164028E-02
  Yp (radians):  6.494762E-02
  Rp (radians):  6.629889E-02
RMS emittances with respect to the z-axis
  EpsiX (pi-m-rad):  4.550165E-04
  EpsiY (pi-m-rad):  1.920744E-03
  EpsiR (pi-m-rad):  1.417018E-03
Maximum displacement from the z-axis
  RMax (meters):  2.567633E-02
Magnitude of the maximum particle angle with respect to the z-axis
  RpMax (radians):  1.341721E-01

```

Note that SI units are used and that quantities are calculated relative to the z axis. The quantities $EpsiX$ and $EpsiY$ are calculated from Eq. 3 and 4 and the new quantity $EpsiR$ is determined from Eq. 5.

GenDist determines statistical quantities using PRT files as input. Table 1 shows an example of output in response to the *Beam analysis* command. There are three differences from the **Trak** analysis:

- Calculations may be performed for all PRT files created in a run.
- Length dimensions are those used in the **Trak** simulation. For example, with $DUnit = 100.0$, emittance dimensions are π -cm-rad.
- Calculations may be performed with respect to the x , y or z axes. As an example, for reference axis y the radial quantity used in Eq. 5 is $r = \sqrt{z^2 + x^2}$.

Test calculations

I set up the **Trak** simulation EMITCALC to check the validity of the new routines and to demonstrate how to use **GenDist** to create input circular beams with non-zero emittance. Figure 1 shows the geometry. The solution volume had outer radius 20.0 cm and axial length 100.0 cm. A low-current beam electron beam was injected through the left-hand boundary. The beam, with radius 2.5 cm and kinetic energy 250 keV, passed through a solenoid lens. The **PerMag** solution for the lens had a peak field of 25.0 G. A

Table 1: **GenDist** statistical report

Beam distribution analysis for file emitcalc03.PRT

Reference axis: Z
X(origin): 0.0000E+00
Y(origin): 0.0000E+00

Averaged quantities in absolute space

Total number of entries in file: 3000
Average position
X: 9.0926E-03
Y: 1.6374E-03
Z: 7.5000E+01
Average direction vector
Ux: -0.0003853
Uy: 0.0001055
Uz: 0.9972631
Average energy: 2.5000E+05 (eV)

Averaged quantities relative to the reference axis

Average displacement
X-XRef: 9.0926E-03
Y-YRef: 1.6374E-03
Average angle (degrees)
ThetaX: -0.022
ThetaY: 0.006
RMS angular spread (degrees)
dThetaX: 2.894
dThetaY: 3.104
RMS position spread relative to reference axis
dX: 7.3704E-01
dY: 7.5249E-01
dR: 1.0533E+00
RMS position spread relative to average position
dX: 7.3698E-01
dY: 7.5248E-01
dR: 1.0533E+00
RMS emittance
EpsiX: 1.3764E-01
EpsiY: 1.5474E-01
EpsiR: 1.4037E-01
RMS energy spread: 3.3975E+00 (eV)

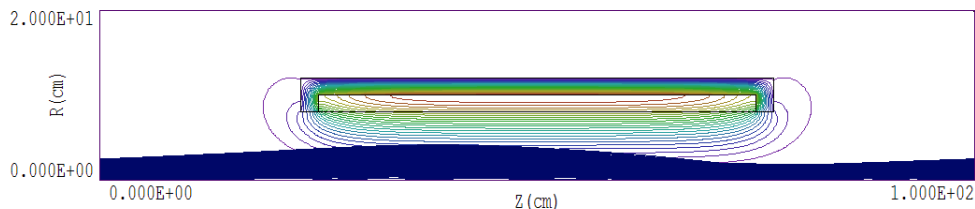


Figure 1: Example EMITCALC: z - r plot of magnetic field lines and model-particle trajectories.

normalization factor of 4.8 was introduced in the **Trak** simulation to give a beam rotation of about 90° . I included an option for a uniform axial electric field to check conservation of normalized momentum. The electric field was set to zero for the reported calculations.

I prepared two **GenDist** input files to create 1) a full circular beam (EMITCALCFULL) and 2) a line beam along the x axis (EMITCALCLINE). Table 2 shows the file to create the full beam. The *Def(Circ)* command created a circular beam moving in z with radius 2.5 cm and 2500 model particles distributed uniformly in azimuth. The weighting function defined in the *RDist* structure ensured an approximately uniform density of particles in configuration space. The *DxDist* and *DyDist* commands introduced a uniform distribution in transverse angles between ± 0.05 radians at all positions. The quantity $\bar{\epsilon}_r$ should be roughly equal to 0.125π -cm-rad.

The **Trak** input file EMITCALC.TIN has the content

```

FIELDS
  EFILE: EmitCalcE.EOU 0.0
  BFILE: EmitCalcB.POU 4.8
  DUNIT: 1.0000E+02
END
PARTICLES TRACK
  PFILE: EmitCalcFull
  RECORD UP Z 25.0 50.0 75.0
END
DIAGNOSTICS
  PARTLIST
  PARTFILE: EmitCalcOut
  PARTDIST
  BSCAN 0.0 0.0 0.0 0.0 0.0 100.0
END
ENDFILE

```

Note the use of the RECORDPLANE command. With inclusion of the input file EMITCALCFULL.PRT and output file EMITCALCOUT.PRT, particle files were

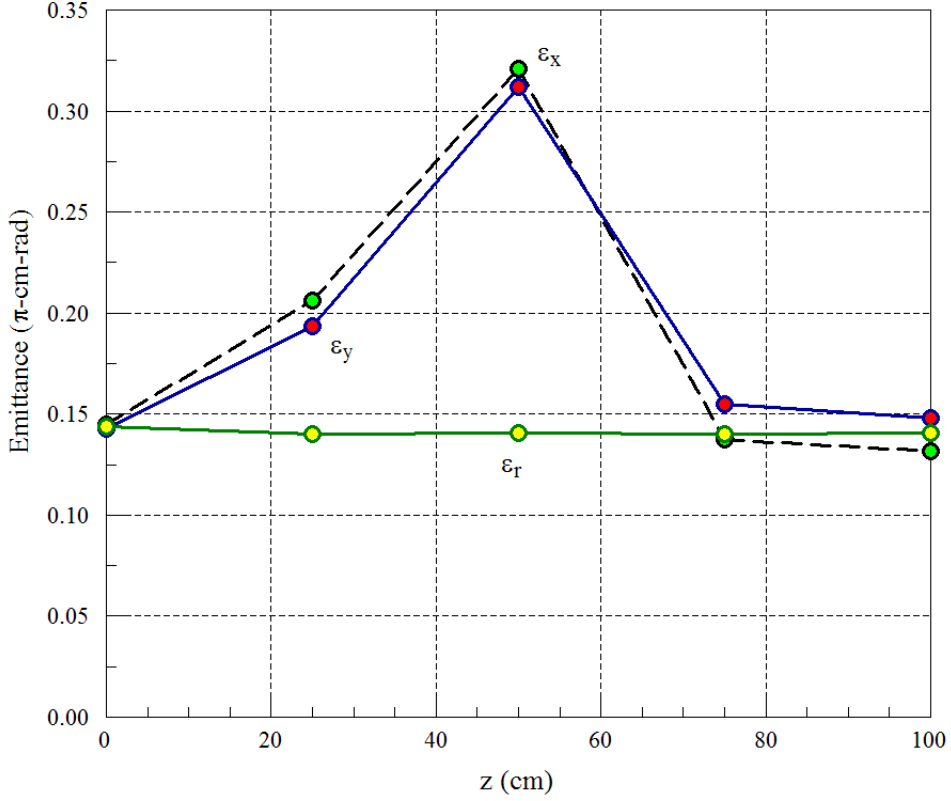


Figure 2: EMITCALC example, azimuthally-symmetric beam. Emittance calculations using **GenDist** at different axial positions.

available at $z = 0.0, 25.0, 50.0, 75.0$ and 100.0 cm. Figure 1 shows the envelope of the high-emittance beam. After expanding from a waist, it was focused to a divergence-limited spot by the lens.

Figure 2 shows calculated values of ϵ_x , ϵ_y and ϵ_r at different axial positions. The condition $\epsilon_x \cong \epsilon_y \cong \epsilon_r$ holds outside the region of the lens field. There were large errors in ϵ_x and ϵ_y in the region where the beam rotates. In contrast, ϵ_r had the fixed value 0.140π -cm-rad throughout the simulation volume.

Next, I prepared an equivalent beam distribution with all particles aligned along x (the typical input condition for a **Trak** simulation). Table 3 shows the **GenDist** file. In this case, the source shape was rectangular with zero width along y . The distribution had one particle along y and 500 particles along x . There was an interesting subtlety in assigning weighting in the x direction. For spatial uniformity, the weighting should follow $p(x) = |x|$. If the rectangular source filled the range $-2.50 \text{ cm} \leq x \leq 2.50 \text{ cm}$, the weighting function would be discontinuous. Instead, I set the source dimensions to $-1.25 \text{ cm} \leq x \leq 1.25 \text{ cm}$, added a shift of 1.25 cm in the x direction, and used

Table 2: Contents of the EMITCALCFULL.DST, **GenDist** input file

```
FileType = PRT
RestMass = 0.0000E+00
Charge = -1.0000E+00
Energy = 2.5000E+05
Def(Circ) = 0.0000E+00 2.5000E+00 50 50
Shift = 0.0000E+00 0.0000E+00 1.0000E-03
Distribution = Uniform
RDist
 0.00 0.00
 0.20 0.20
 0.40 0.40
 0.60 0.60
 0.80 0.80
 1.00 1.00
End
DxDist
 -2.865 1.000
 -2.000 1.000
 -1.000 1.000
 0.000 1.000
 1.000 1.000
 2.000 1.000
 2.865 1.000
End
DyDist
 -2.865 1.000
 -2.000 1.000
 -1.000 1.000
 0.000 1.000
 1.000 1.000
 2.000 1.000
 2.865 1.000
End
EndFile
```

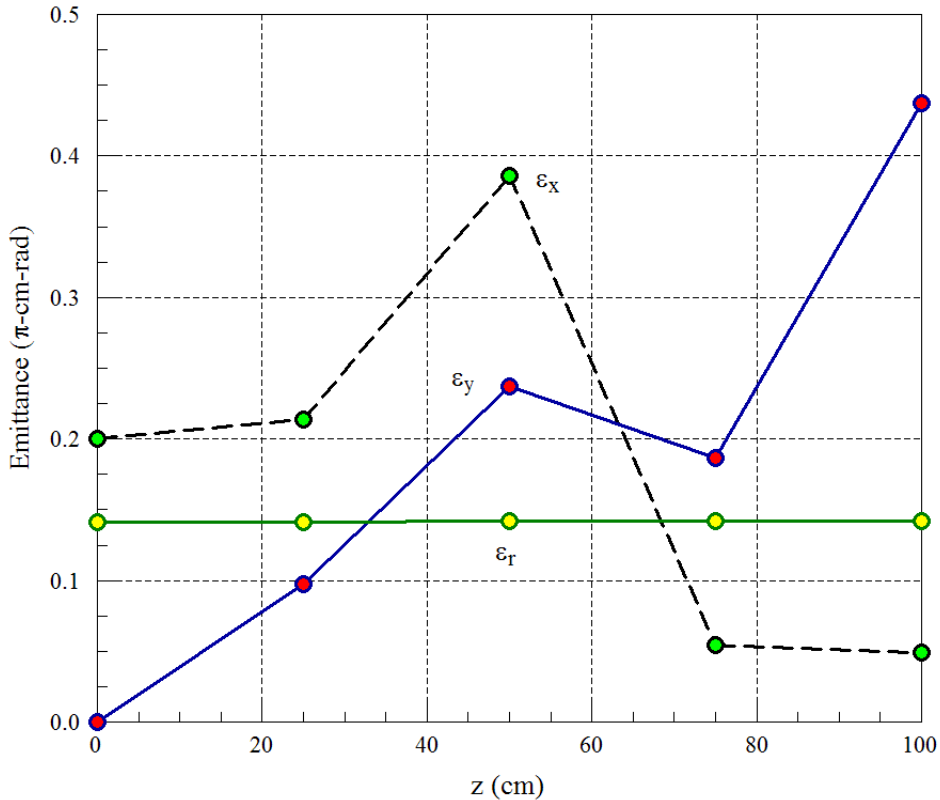


Figure 3: EMITCALC example, input beam aligned along x . Emittance calculations using **GenDist** at different axial positions.

the spatial weighting function $p(x) = x$. The angular distribution in x and y was the same as that for the full beam.

Figure 3 shows the calculated emittances. The quantities ϵ_x and ϵ_y were effectively meaningless over the full solution space. In contrast, ϵ_r was conserved through the lens and had the same value (0.140π -cm-rad) as that for the azimuthally-symmetric beam. In conclusion, the Lee-Cooper emittance expression provides a reliable indicator of radial emittance in **Trak** simulations of circular beams for realistic beam distributions. By realistic, I mean that the beam varies smoothly in space and has an isotropic distribution in angle. Some input distributions lead to unexpected values of ϵ_r because they violate the basis of the RMS emittance definition. The user should avoid singular distributions, such as particles on the boundary of a phase ellipse or a beam with divergence only in one transverse direction.

Table 3: Contents of EMITCALCLINE.DST, GenDist input file

```
FileType = PRT
RestMass = 0.0000E+00
Charge = -1.0000E+00
Energy = 2.5000E+05
Def(Rect) = 1.2500E+00 0.0000E+00 500 1
Shift = 1.2500E+00 0.0000E+00 1.0000E-03
Distribution = Uniform
XDist
  0.00 0.00
  0.20 0.20
  0.40 0.40
  0.60 0.60
  0.80 0.80
  1.00 1.00
End
DxDist
  -2.865 1.000
  -2.000 1.000
  -1.000 1.000
  0.000 1.000
  1.000 1.000
  2.000 1.000
  2.865 1.000
End
DyDist
  -2.865 1.000
  -2.000 1.000
  -1.000 1.000
  0.000 1.000
  1.000 1.000
  2.000 1.000
  2.865 1.000
End
EndFile
```